

Cohomology of n -ary-Nambu-Lie superalgebras and super w_∞ 3-algebra

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Abstract

The purpose of this paper is define the representation and the cohomology of n -ary-Nambu-Lie superalgebras. Moreover we study central extensions and provide as application the computations of the derivations and second cohomology group of super w_∞ 3-algebra.

Introduction

The first instances of n -ary algebras in Physics appeared with a generalization of the Hamiltonian mechanics proposed in 1973 by Nambu [5]. More recent motivation comes from string theory and M-branes involving naturally an algebra with ternary operation called Bagger-Lambert algebra which gives impulse to a significant development. It was used in [6] as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and an $SO(8)$ R-symmetry that acts on the eight transverse scalars. On the other hand in the study of supergravity solutions describing M2-branes ending on M5-branes, the Lie algebra appearing in the original Nahm equations has to be replaced with a generalization involving ternary bracket in the lifted Nahm equations, see [7]. For other applications in Physics see [8, 9, 10].

The algebraic formulation of Nambu mechanics is due to Takhtajan [11, 19] while the abstract definition of n -ary Nambu algebras or n -ary Nambu-Lie algebras (when the bracket is skew symmetric) was given by Filippov in 1985 see [13]. The Leibniz n -ary algebras were introduced and studied in [14]. For deformation theory and cohomologies of n -ary algebras of Lie type, we refer to [15, 16, 17, 18, 19].

The paper is organized as follows. In the first section we give the definitions and some key constructions of n -ary-Nambu-Lie superalgebras. In Section 2 we define a derivation of n -ary-Nambu-Lie superalgebra. Section 3 is dedicated to the representation theory n -ary-Nambu-Lie superalgebras including adjoint representation. In Section 3 we construct family of cohomologies

of n -ary-Nambu-Lie superalgebras. In Section 4, we discuss extensions of n -ary-Nambu-Lie superalgebras and their connection to cohomology. In the last section we compute the derivations and cohomology group of the super w_∞ 3-algebra.

1 The n -ary-Nambu-Lie superalgebra

Let \mathcal{N} be a linear superspace over a field \mathbb{K} that is a \mathbb{Z}_2 -graded linear space with a direct sum $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$.

The elements of \mathcal{N}_j , $j \in \mathbb{Z}_2$, are said to be homogenous of parity j . The parity of a homogeneous element x is denoted by $|x|$.

Let x_1, \dots, x_n be n homogenous elements of \mathcal{N} , we denote by $|(x_1, \dots, x_n)| = |x_1| + \dots + |x_n|$ the parity of an element (x_1, \dots, x_n) in \mathcal{N}^n .

The space $End(\mathcal{N})$ is \mathbb{Z}_2 graded with a direct sum $End(\mathcal{N}) = (End(\mathcal{N}))_0 \oplus (End(\mathcal{N}))_1$ where $(End(\mathcal{N}))_j = \{f \in End(\mathcal{N})/f(\mathcal{N}_i) \subset \mathcal{N}_{i+j}\}$. The elements of $(End(\mathcal{N}))_j$ are said to be homogenous of parity j .

Definition 1.1. [2] An n -ary-Nambu superalgebra is a pair $(\mathcal{N}, [., \dots, .])$ consisting of a vector superspace \mathcal{N} and even n -linear map $[., \dots, .] : \mathcal{N}^n \rightarrow \mathcal{N}$, satisfying

$$\left[y_2, \dots, y_n, [x_1, \dots, x_n] \right] = \sum_{i=1}^n (-1)^{(|y_2| + \dots + |y_n|)(|x_1| + \dots + |x_{i-1}|)} \left[x_1, \dots, [y_2, \dots, y_n, x_i], \dots, x_n \right] \quad (1.1)$$

for all $(x_1, \dots, x_n) \in \mathcal{N}^n$, $(y_2, \dots, y_n) \in \mathcal{N}^{n-1}$.

The identity (1.1) is called Nambu identity.

Definition 1.2. An n -ary-Nambu superalgebra $(\mathcal{N}, [., \dots, .])$ is called n -ary-Nambu-Lie superalgebra if the bracket is skew-symmetric that is

$$[x_1, \dots, x_{i-1}, x_i, \dots, x_n] = -(-1)^{|x_{i-1}||x_i|} [x_1, \dots, x_i, x_{i-1}, \dots, x_n]. \quad (1.2)$$

Definition 1.3. Let $(\mathcal{N}, [., \dots, .])$ and $(\mathcal{N}', [., \dots, .]')$ be two n -ary-Nambu-Lie superalgebra. An homomorphism $f : \mathcal{N} \rightarrow \mathcal{N}'$ is said to be morphism of n -ary-Nambu-Lie superalgebra if

$$[f(x_1), \dots, f(x_n)]' = f([x_1, \dots, x_n]) \quad \forall x_1, \dots, x_n \in \mathcal{N}. \quad (1.3)$$

Proposition 1.4. Let f be an even endomorphism of n -ary-Nambu-Lie superalgebra $(\mathcal{N}, [., \dots, .])$.

We can define the new n -ary-Nambu-Lie superalgebra $(\mathcal{N}, [., \dots, .]')$, where $[x_1, \dots, x_n]' = f([x_1, \dots, x_n])$.

2 Derivation of n -ary-Nambu-Lie superalgebra

Definition 2.1. [2] We call $D \in End_i(\mathcal{N})$, where i is in \mathbb{Z}_2 , a derivation of the n -ary-Nambu-Lie superalgebra $(\mathcal{N}, [., \dots, .])$ if

$$D([x_1, \dots, x_n]) = \sum_{k=1}^n (-1)^{|D|(|x_1| + \dots + |x_{k-1}|)} [x_1, \dots, D(x_k), \dots, x_n], \quad \text{for all homogeneous } x_1, \dots, x_n \in \mathcal{N}.$$

We denote by $Der(\mathcal{N}) = Der_{\overline{0}}(\mathcal{N}) \oplus Der_{\overline{1}}(\mathcal{N})$ the set of derivation of the n -ary-Nambu-Lie superalgebra \mathcal{N} .

The subspace $Der(\mathcal{N}) \subset End(\mathcal{N})$ is easily seen to be closed under the bracket

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1 \quad (2.1)$$

(known as the supercommutator) and it is called the superalgebra of derivations of \mathcal{N} .

With above notation, $Der(\mathcal{N})$ is a Lie superalgebra, in wich the Lie bracket is given by (2.1).

Fix $n-1$ homogeneous elements $x_1, \dots, x_{n-1} \in \mathcal{N}$, and define the transformations $ad(x_1, \dots, x_{n-1}) \in End(\mathcal{N})$ by the rule

$$ad(x_1, \dots, x_{n-1})(x) = [x_1, \dots, x_{n-1}, x]. \quad (2.2)$$

Then $ad(x_1, \dots, x_{n-1})$ is a derivation of \mathcal{N} , wich we call inner derivation of \mathcal{N} .

Indeed we have

$$\begin{aligned} ad(y_2, \dots, y_n)([x_1, \dots, x_n]) &= [y_2, \dots, y_n, [x_1, \dots, x_n]] \\ &= \sum_{i=1}^n (-1)^{(|y_2|+\dots+|y_n|)(|x_1|+\dots+|x_{i-1}|)} [x_1, \dots, ad(y)(x_i), \dots, x_n] \\ &= \sum_{i=1}^n (-1)^{|ad(y)|(|x_1|+\dots+|x_{i-1}|)} [x_1, \dots, ad(y)(x_i), \dots, x_n]. \end{aligned}$$

3 Representations of n -ary-Nambu-Lie superalgebra

We provide in the following a graded version of the study of representations of n -ary-Nambu-Lie algebra stated in [3].

Let $(\mathcal{N}, [., \dots, .])$ be a n -ary-Nambu-Lie superalgebra and $V = V_{\overline{0}} \oplus V_{\overline{1}}$ an arbitrary vector superspace. Let $[., .]_V : \mathcal{N}^{n-1} \times V \longrightarrow V$ be a bilinear map satisfying $[\mathcal{N}_i^{n-1}, V_j]_V \subset V_{i+j}$ where $i, j \in \mathbb{Z}_2$.

Definition 3.1. The pair $(V, [., .]_V)$ is called a module on the n -ary-Nambu-Lie superalgebra $\mathcal{N} = \mathcal{N}_{\overline{0}} \oplus \mathcal{N}_{\overline{1}}$ or \mathcal{N} -module V if the even multilinear mapping $[., \dots, .]_V$ satisfies

$$\begin{aligned} & [ad(x)(x_n), y_2, \dots, y_{n-1}, v]_V \\ &= \sum_{i=1}^n (-1)^{n-i+|x_i|(|x_{i+1}|+\dots+|x_n|)} [x_1, \dots, \widehat{x_i} \dots, x_n, [x_i, y_2, \dots, y_{n-1}, v]_V]_V, \\ & \sum_{i=1}^{n-1} (-1)^{|y|(|x_1|+\dots+|x_{i-1}|)} [x_1, \dots, ad(y)(x_i), \dots, x_{n-1}, v]_V = [y, [x, v]_V]_V - (-1)^{|y||x|} [x, [y, v]_V]_V \end{aligned} \quad (3.1)$$

for all homogeneous x, y in \mathcal{N}^{n-1} and v in V . It will also say that $(V, [., \dots, .]_V)$ is a representation of \mathcal{N} .

Example 3.2. Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and the map ad defined in (2.2). Then (\mathcal{N}, ad) is a representation of \mathcal{N} .

Remark 3.3. When $[\cdot, \cdot]_V$ is the zero map, we say that the module V is trivial.

4 Cohomology of n -ary-Nambu-Lie superalgebra induced by cohomology of Leibniz algebras

In this section, we aim to extend to n -ary Nambu superalgebra type of process introduced by Takhtajan to construct a complex of n -ary Nambu superalgebra starting from a complex of binary algebras (see [20]).

Definition 4.1. A Leibniz superalgebra is a pair $(L, [\cdot, \cdot])$ consisting of a vector superspace L and bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ satisfying

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]]. \quad (4.1)$$

Let $(L, [\cdot, \cdot])$ be a Leibniz superalgebra and W be an arbitrary vector superspace. Let $[\cdot, \cdot]_W : L \times W \rightarrow W$ be an even bilinear map satisfying

$$[[x, y], w]_W = [x, [y, w]_W]_W - (-1)^{|x||y|} [y, [x, w]_W]_W.$$

The pair $(W, [\cdot, \cdot]_W)$ is called an L -module.

Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and $(V, [\cdot, \dots, \cdot]_V)$ be a \mathcal{N} -module. We denote by $\mathcal{L}(\mathcal{N})$ the space $\wedge^{n-1} \mathcal{N}$ and we call it the fundamental set. We define a bilinear map $[\cdot, \cdot]_L : \mathcal{L}(\mathcal{N}) \times \mathcal{L}(\mathcal{N}) \rightarrow \mathcal{L}(\mathcal{N})$ and $ad : \mathcal{N} \times V \rightarrow V$ respectively by

$$[x, y]_L = \sum_{i=1}^{n-1} (-1)^{|x|(|y_1| + \dots + |y_{i-1}|)} y_1 \wedge \dots \wedge ad(x)(y_i) \wedge \dots \wedge y_{n-1} \quad (4.2)$$

and

$$ad(x, v) = [x, v]_V \quad (4.3)$$

for all $x = x_1 \wedge \dots \wedge x_{n-1}$, $y = y_1 \wedge \dots \wedge y_{n-1} \in \mathcal{L}(\mathcal{N})$, $v \in V$.

Lemma 4.2. Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and be V be a \mathcal{N} -module. The map ad satisfies

$$ad([x, y]_L)(v) = ad(x)(ad(y)(v)) - (-1)^{|x||y|} ad(y)(ad(x)(v)) \quad (4.4)$$

for all $x, y \in \mathcal{L}(\mathcal{N})$, $v \in V$.

Proof. By (4.3) and (3.2) we have

$$\begin{aligned}
ad([x, y]_L)(v) &= [[x, y]_L, v]_V \\
&= \sum_{i=1}^{n-1} (-1)^{|x|(|y_1|+\dots+|y_{i-1}|)} [y_1 \wedge \dots \wedge ad(x)(y_i) \wedge \dots \wedge y_{n-1}, v]_V. \\
&= [x_1, \dots, x_{n-1}, [y_1, \dots, y_{n-1}, v]]_V - (-1)^{|x|(|y_1|+\dots+|y_{n-1}|)} [y_1, \dots, y_{n-1}, [x_1, \dots, x_{n-1}, v]]_V \\
&= ad(x)(ad(y)(v)) - (-1)^{|x||y|} ad(y)(ad(x)(v)).
\end{aligned}$$

□

Proposition 4.3. *The pair $(\mathcal{L}(\mathcal{N}), [\cdot, \cdot]_L)$ is a Leibniz superalgebra.*

Let $(V, [\cdot, \dots, \cdot]_V)$ be a \mathcal{N} -module. We denote

$$W = \mathcal{L}(\mathcal{N}, V) = \{x_1 \wedge \dots \wedge x_{n-2} \wedge v, x_i \in \mathcal{N}, v \in V\}.$$

Note that

$$u_1 \wedge \dots \wedge u_{j-1} \wedge u_j \wedge \dots \wedge u_{n-1} = -(-1)^{|u_{j-1}||u_j|} u_1 \wedge \dots \wedge u_i \wedge u_{i-1} \wedge \dots \wedge u_{n-1}, \quad (4.5)$$

for all homogenous element $u = u_1 \wedge \dots \wedge u_{n-1}$ of W .

Define a bilinear map $[\cdot, \cdot]_W : \mathcal{L}(\mathcal{N}) \longrightarrow W$ by

$$\begin{aligned}
[x, y_1 \wedge \dots \wedge y_1 \wedge v]_W &= \sum_{i=1}^{n-1} (-1)^{|x|(|y_1|+\dots+|y_{i-1}|)} y_1 \wedge \dots \wedge ad(x)(y_i) \wedge \dots \wedge y_{n-1} \wedge v \\
&\quad + (-1)^{|x|(|y_1|+\dots+|y_{n-1}|)} y_1 \wedge \dots \wedge y_{n-1} \wedge ad(x)(v).
\end{aligned}$$

Proposition 4.4. *The pair $(W, [\cdot, \cdot]_W)$ is a L -module.*

In the following, the expression $[x, y]$ means:

- $[x, y]_L$ if $x, y \in \mathcal{L}(\mathcal{N})$.
- $[x_1, \dots, x_{n-1}, y]$ if $x = x_1 \wedge \dots \wedge x_{n-1} \in \mathcal{L}(\mathcal{N}), y \in \mathcal{N}$.

Definition 4.5. We call k -cochain of a n -ary super-algebra \mathcal{N} with values in V a multilinear map

$$\varphi : \mathcal{L}(\mathcal{N})^k \times \mathcal{N} \longrightarrow V.$$

Denote $C^k(\mathcal{N}, V)$ the set of k -cochains on \mathcal{N} with values in V .

Theorem 4.6. *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and $C^k(\mathcal{L}(\mathcal{N}), V)$ the set of k -cochain.*

We define a coboundary operator $d^k : C^k(\mathcal{L}(\mathcal{N}), W) \rightarrow C^{k+1}(\mathcal{L}(\mathcal{N}), W)$ by $df(x) = -[x, f]$ when $f \in C^0(\mathcal{L}(\mathcal{N}), V) = V$ and for $k \geq 1$,

$$\begin{aligned} d^k(f)(x_0, \dots, x_k) &= - \sum_{0 \leq s < t \leq k} (-1)^{s+|x_s|(|x_{s+1}|+\dots+|x_{t-1}|)} f(x_0, \dots, \widehat{x}_s, \dots, x_{t-1}, [x_s, x_t], x_{t+1}, \dots, x_k, z) \\ &\quad + \sum_{s=0}^{k-1} (-1)^{s+|x_s|(|f|+|x_0|+\dots+|x_{s-1}|)} \left[x_s, f(x_0, \dots, \widehat{x}_s, \dots, x_k) \right]_W \\ &\quad + (-1)^k \left[f(x_0, \dots, x_{k-1}), x_k \right]_W', \end{aligned}$$

where

$$\begin{aligned} &\left[x_1 \wedge \dots \wedge x_{n-2} \wedge v, y_1 \wedge \dots \wedge y_{n-1} \right]_W' \\ &= - \sum_{i=1}^{n-1} (-1)^{(|x_1|+\dots+|x_{n-2}|+|v|)(|y_1|+\dots+|y_{i-1}|)+|v||y_i|} y_1 \wedge \dots \wedge ad(x_1 \wedge \dots \wedge x_{n-2} \wedge y_i)(v) \wedge \dots \wedge y_{n-1} \end{aligned}$$

Let $\Delta^k : C^{k-1}(\mathcal{N}, V) \rightarrow C^k(\mathcal{L}(\mathcal{N}), W)$ be the linear map defined for $k = 0$ by

$$\Delta(f)(x_0) = \sum_{i=1}^{n-1} (-1)^{|f|(|x_0^1|+\dots+|x_0^{i-1}|)} x_0^1 \wedge \dots \wedge f(x_0^i) \wedge \dots \wedge x_0^{n-1}$$

and for $k > 0$ by

$$\begin{aligned} &\Delta^k(f)(x_0, \dots, x_k) \\ &= \sum_{i=1}^{n-1} (-1)^{(|f|+|x_0|+\dots+|x_{k-1}|)(|x_k^1|+\dots+|x_k^{i-1}|)} x_k^1 \wedge \dots \wedge f(x_0, \dots, x_{k-1}, x_k^i) \wedge x_k^{i+1} \wedge \dots \wedge x_k^{n-1} \end{aligned}$$

where we set $x_j = x_j^1 \wedge \dots \wedge x_j^{n-1}$. Then there exists a cohomology complex $(C^k(\mathcal{N}, \mathcal{N}), \delta)$ for n -ary-Nambu-Lie superalgebra such that

$$d^k \circ \Delta^{k-1} = \Delta^{k-1} \circ \delta^{k-1}$$

The coboundary map $\delta^{k+1} : C^k(\mathcal{N}, V) \rightarrow C^{k+1}(\mathcal{N}, V)$ is defined by

$$\begin{aligned} &\delta^{k+1}(f)(x_0, \dots, x_k, z) \\ &= - \sum_{0 \leq s < t \leq k} (-1)^{s+|x_s|(|x_{s+1}|+\dots+|x_{t-1}|)} f(x_0, \dots, \widehat{x}_s, \dots, x_{t-1}, [x_s, x_t], x_{t+1}, \dots, x_k, z) \\ &\quad - \sum_{s=0}^k (-1)^{s+|x_s|(|x_{s+1}|+\dots+|x_k|)} f\left(x_0, \dots, x_{s-1}, x_{s+1}, \dots, x_k, ad(x_s)(z)\right) \\ &\quad + \sum_{s=0}^k (-1)^{s+|x_s|(|f|+|x_0|+\dots+|x_{s-1}|)} \left[x_s, f(x_0, \dots, \widehat{x}_s, \dots, x_k, z) \right]_V. \\ &\quad + \sum_{i=1}^{n-1} (-1)^{k-i+(|f|+|x_0|+\dots+|x_{k-1}|+|x_k^{i+1}|+\dots+|x_k^{n-1}|)(|x_k^i|+|x_k|)+|z|(|f|+|x_0|+\dots+|x_k|)+|x_k|(|x_k^{i+1}|+\dots+|x_k^{n-1}|)} \\ &\quad \left[z \wedge x_k^1 \wedge \dots \wedge \widehat{x}_k^i \wedge \dots \wedge x_k^{n-1}, f\left(x_0, \dots, x_{k-1}, x_k^i\right) \right]_V. \end{aligned}$$

Proof. For any $f \in C^{k-2}(\mathcal{N}, V)$, by calculation we obtain $d^{k+1} \circ d^k(f)(x_0, \dots, x_k) = 0$ and $d^k \circ \Delta^{k-1}(f)(x_0, \dots, x_k) = \Delta^{k-1} \circ \delta^{k-1}(f)(x_0, \dots, x_k)$.

One has $\Delta^{k+1} \circ \delta^k = d^k \circ \Delta^k$, then $\Delta^{k+1} \circ \delta^k \circ \delta^{k-1} = d^k \circ d^{k-1} \circ \Delta^{k-1} = 0$, because $d^k \circ d^{k-1} = 0$ \square

Definition 4.7. • The k -cocycles space is defined as $Z^k(\mathcal{N}, V) = \ker \delta^k$. The even (resp. odd) k -cocycles space is defined as $Z^k(\mathcal{N}, V)_0 = Z^k(\mathcal{N}, V) \cap (C^k(\mathcal{N}, V))_0$ (resp. $Z^k(\mathcal{N}, V)_1 = Z^k(\mathcal{N}, V) \cap (C^k(\mathcal{N}, V))_1$).

- The k -coboundaries space is defined as $B^k(\mathcal{N}, V) = \text{Im } \delta^{k-1}$. The even (resp. odd) k -coboundaries space is $B_0^k(\mathcal{N}, V) = B^k(\mathcal{N}, V) \cap (C^k(\mathcal{N}, V))_0$ (resp. $B_1^k(\mathcal{N}, V) = B^k(\mathcal{N}, V) \cap (C^k(\mathcal{N}, V))_1$).
- The k^{th} cohomology space is the quotient $H^k(\mathcal{N}, V) = Z^k(\mathcal{N}, V)/B^k(\mathcal{N}, V)$. It decomposes as well as even and odd k^{th} cohomology spaces.

Finally, we denote by $H^k(\mathcal{N}, V) = H_0^k(\mathcal{N}, V) \oplus H_1^k(\mathcal{N}, V)$ the k^{th} cohomology space and by $\oplus_{k \geq 0} H^k(\mathcal{N}, V)$ the r -cohomology group of the Hom-Lie superalgebra \mathcal{N} with values in V .

Remark 4.8. The subspace $Z^1(\mathcal{N}, \mathcal{N})$ is the set of derivation of \mathcal{N} .

5 Extensions of n-ary-Nambu-Lie superalgebra

An extension theory of Hom-Lie superalgebras was stated in [4].

An extension of a n -ary-Nambu-Lie superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot])$ by \mathcal{N} -module $(V, [\cdot, \dots, \cdot]_V)$ is an exact sequence

$$0 \longrightarrow (V, [\cdot, \dots, \cdot]_V) \xrightarrow{i} (\widetilde{\mathcal{N}}, [\cdot, \dots, \cdot]) \xrightarrow{\pi} (\mathcal{N}, [\cdot, \dots, \cdot]) \longrightarrow 0.$$

We say that the extension is central if $[\mathcal{L}(\widetilde{\mathcal{N}}), i(V)]_{\widetilde{\mathcal{N}}} = 0$.

Two extensions

$$0 \longrightarrow (V, [\cdot, \dots, \cdot]_V) \xrightarrow{i_k} (\widetilde{\mathcal{N}}_k, [\cdot, \dots, \cdot]) \xrightarrow{\pi_k} (\mathcal{N}, [\cdot, \dots, \cdot]) \longrightarrow 0 \quad (k = 1, 2)$$

are equivalent if there is an isomorphism $\varphi : (\mathcal{N}_1, [\cdot, \dots, \cdot]_1) \longrightarrow (\mathcal{N}_2, [\cdot, \dots, \cdot]_2)$ such that $\varphi \circ i_1 = i_2$ and $\pi_2 \circ \varphi = \pi_1$.

Proposition 5.1. *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and V be a \mathcal{N} -module. The second cohomology space $H^2(\mathcal{N}, V) = Z^2(\mathcal{N}, V)/B^2(\mathcal{G}, V)$ is in one-to-one correspondence with the set of the equivalence classes of central extensions of $(\mathcal{N}, [\cdot, \dots, \cdot])$ by $(V, [\cdot, \dots, \cdot]_V)$.*

Proof. Let

$$0 \longrightarrow (V, [\cdot, \dots, \cdot]_V) \xrightarrow{i} (\widetilde{\mathcal{N}}, [\cdot, \dots, \cdot]) \xrightarrow{\pi} (\mathcal{N}, [\cdot, \dots, \cdot]) \longrightarrow 0.$$

be a central extension of n -ary-Nambu-Lie superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot])$ by $(V, [\cdot, \dots, \cdot]_V)$, so there is a space H such that $\widetilde{\mathcal{N}} = H \oplus i(V)$.

The map $\pi_{/H} : H \rightarrow \mathcal{N}$ (resp $k : V \rightarrow i(V)$) defined by $\pi_{/H}(x) = \pi(x)$ (resp. $k(v) = i(v)$) is bijective, its inverse s (resp. l) note. Considering the map $\varphi : \mathcal{N} \times V \rightarrow \tilde{\mathcal{N}}$ defined by $\varphi(x, v) = s(x) + i(v)$, it is easy to verify that φ is a bijective.

For all $x = x_1 \wedge \dots \wedge x_{n-1} \in \mathcal{L}(\mathcal{N})$ (resp. $v = v_1 \wedge \dots \wedge v_{n-1} \in \mathcal{L}(V)$) we denote $s(x_1) \wedge \dots \wedge s(x_{n-1})$ (resp. $i(v_1) \wedge \dots \wedge i(v_{n-1})$) by $s(x)$. (resp. $i(v)$).

Since π is homomorphism of n -ary-Nambu-Lie superalgebra then $\pi([s(x), s(z)]_{\tilde{\mathcal{N}}} - s([x, z])) = 0$ so $[s(x), s(z)]_{\tilde{\mathcal{N}}} - s([x, z]) \in i(V)$.

We set $[s(x), s(z)] - s([x, z]) = G(x, z) \in i(V)$ then $F(x, z) = l \circ G(x, z) \in V$, it easy to see that $F \in C^2(\mathcal{N}, V)$ is a 2-cochain that defines a bracket on $\tilde{\mathcal{N}}$. In fact, we can identify as a superspace $\mathcal{N} \times V$ and $\tilde{\mathcal{N}}$ by $\varphi : (x, v) \rightarrow s(x) + i(v)$ where the bracket is

$$[s(x) + i(v), s(z) + i(w)]_{\tilde{\mathcal{N}}} = [s(x), s(z)]_{\tilde{\mathcal{N}}} = s([x, z]) + F(x, z).$$

Viewed as elements of $\mathcal{N} \times V$ we have $[(x, v), (z, w)] = ([x, z], F(x, z))$ and the homogeneous elements (x, v) of $\mathcal{N} \times V$ are such that $|x| = |v|$ and we have in this case $|(x, v)| = |x|$.

We deduce that for every central extension

$$0 \longrightarrow (V, [., \dots, .]_V) \xrightarrow{i} (\tilde{\mathcal{N}}, [\widetilde{., \dots, .}]) \xrightarrow{\pi} (\mathcal{N}, [., \dots, .]) \longrightarrow 0.$$

One may associate a two cocycle $F \in Z^2(\mathcal{N}, V)$. Indeed, for $x \in \mathcal{L}(\mathcal{N}), z \in \mathcal{N}$, if we set

$$F(x, z) = l([s(x), s(z)] - s([x, z])) \in V,$$

then, we have $F(x, z) \in V$ and F satisfies the 2-cocycle conditions.

Conversely, for each $f \in Z^2(\mathcal{N}, V)$, one can define a central extension

$$0 \longrightarrow (V, [., \dots, .]_V) \longrightarrow (\mathcal{N}_f, [., \dots, .]_f) \longrightarrow (\mathcal{N}, [., \dots, .]) \longrightarrow 0,$$

by

$$[(x, v), (y, w)]_f = ([x, y], f(x, y)),$$

where $x \in \mathcal{L}(\mathcal{N}), z \in \mathcal{N}$ and $v \in \mathcal{L}(V), w \in V$.

Let f and g be two elements of $Z^2(\mathcal{N}, V)$ such that $f - g \in B^2(\mathcal{G}, V)$ i.e. $(f - g)(x, z) = h([x, z])$, where $h : \mathcal{N} \rightarrow V$ is a linear map. Now we prove that the extensions defined by f and g are equivalent. Let us define $\Phi : \mathcal{N}_f \times V \rightarrow \mathcal{N}_g \times V$ by

$$\Phi(x, v) = (x, v - h(x)).$$

It is clear that Φ is bijective. Let us check that Φ is a homomorphism of n -ary-Nambu-Lie superalgebra. We have

$$\begin{aligned} [\Phi((x, v)), \Phi((z, w))]_g &= [(x, v - h(x)), (z, w - h(z))]_g \\ &= ([x, z], g(x, z)) \\ &= ([x, z], f(x, z) - h([x, z])) \\ &= \Phi([x, z], f(x, z)) \\ &= \Phi([(x, v), (z, w)]_f). \end{aligned}$$

Next, we show that for $f, g \in Z^2(\mathcal{N}, V)$ such that the central extensions

$$0 \rightarrow (V, [\cdot, \dots, \cdot]_V) \rightarrow (\mathcal{N}_f, [\cdot, \dots, \cdot]_f) \rightarrow (\mathcal{N}, [\cdot, \dots, \cdot]) \rightarrow 0, \text{ and}$$

$0 \rightarrow (V, [\cdot, \dots, \cdot]_V) \rightarrow (\mathcal{N}_g, [\cdot, \dots, \cdot]_g) \rightarrow (\mathcal{N}, [\cdot, \dots, \cdot]) \rightarrow 0$, are equivalent, we have $f - g \in B^2(\mathcal{N}, V)$. Let Φ be a homomorphism of n-ary-Nambu-Lie superalgebra . such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & (V, [\cdot, \dots, \cdot]_V) & \xrightarrow{i_1} & (\mathcal{N}_f, [\cdot, \dots, \cdot]_f) & \xrightarrow{\pi_1} & (\mathcal{N}, [\cdot, \dots, \cdot]) \longrightarrow 0 \\ & & \downarrow id_V & & \downarrow \Phi & & \downarrow id_{\mathcal{N}} \\ 0 & \longrightarrow & (V, [\cdot, \dots, \cdot]_V) & \xrightarrow{i_2} & (\mathcal{N}_g, [\cdot, \dots, \cdot]_g) & \xrightarrow{\pi_2} & (\mathcal{N}, [\cdot, \dots, \cdot]) \longrightarrow 0 \end{array}$$

commutes. We can express $\Phi(x, v) = (x, v - h(x))$ for some linear map $h : \mathcal{N} \rightarrow V$. Then we have

$$\begin{aligned} \Phi([(x, v), (z, w)]_f) &= \Phi([(x, z], f(x, z))) \\ &= ([x, z], f(x, z) - h([x, z])), \end{aligned}$$

$$\begin{aligned} [\Phi((x, v)), \Phi((z, w))]_g &= [(x, v - h(x)), (z, w - h(y))]_g \\ &= ([x, z], g(x, z)), \end{aligned}$$

and thus $(f - g)(x, y) = h([x, z])$ (i.e. $f - g \in B^2(\mathcal{N}, V)$), so we have completed the proof. \square

5.1 Deformation of n-ary-Nambu-Lie superalgebra.

Definition 5.2. Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n-ary-Nambu-Lie superalgebra. A one -parameter formal Lie super deformation of \mathcal{N} is given by the $\mathbb{K}[[t]]$ -multilinear map $[\cdot, \dots, \cdot]_t : \mathcal{N}^{n-1}[[t]] \times \mathcal{N}[[t]] \rightarrow \mathcal{N}[[t]]$ of the form

$$[\cdot, \dots, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \dots, \cdot]_i,$$

where each $[\cdot, \dots, \cdot]_i$ is a even multiilinear map $[\cdot, \dots, \cdot]_i : \mathcal{N}^{n-1} \times \mathcal{N} \rightarrow \mathcal{N}$ (extended to be $\mathbb{K}[[t]]$ -multilinear), $[\cdot, \dots, \cdot] = [\cdot, \dots, \cdot]_0$ and satisfying the following conditions

$$[x_1, \dots, x_i, \dots, x_j, \dots, x_n]_t = (-1)^{j-i+|x_i|(|x_{i+1}|+\dots+|x_{j-1}|)} [x_1, \dots, x_j, \dots, x_i, \dots, x_n]_t, \quad (5.1)$$

$$[x, [y, z]_t]_t - [[x, y]_t, z]_t - (-1)^{|y||x|} [y, [x, z]_t]_t = 0 \quad (5.2)$$

for all homogeneous elements $x, y \in \mathcal{N}^{n-1}$ and $z \in \mathcal{N}$. The super deformation is said to be of

order k if $[\cdot, \dots, \cdot]_t = \sum_{i=0}^k t^i [\cdot, \dots, \cdot]_i$.

Given two deformation $\mathcal{N}_t = (\mathcal{N}, [\cdot, \dots, \cdot]_t)$ and $\mathcal{N}'_t = (\mathcal{N}', [\cdot, \dots, \cdot]'_t)$ of \mathcal{N} where $[\cdot, \dots, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \dots, \cdot]_i$ and $[\cdot, \dots, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \dots, \cdot]'_i$ with $[\cdot, \dots, \cdot]_0 = [\cdot, \cdot]'_0 = [\cdot, \dots, \cdot]$. We say that \mathcal{N}_t and

\mathcal{N}'_t are equivalent if there exists a formal automorphism $\Phi_t = \sum_{i=0}^k \Phi_i t^i$ where $\Phi_i \in \text{End}(\mathcal{N})$ and $\Phi_0 = id_{\mathcal{N}}$, such that

$$\Phi_t([x, z]_t) = [\Phi_t(x), \Phi_t(z)]'_t.$$

A deformation \mathcal{N}_t is said to be trivial if and only if \mathcal{N}_t is equivalent to \mathcal{N} (viewed as superalgebra on $\mathcal{N}[[t]]$).

The identity (5.2) is called a deformation equation and it is equivalent to

$$\sum_{i \geq 0, j \geq 0} \left([x, [y, z]_i]_j - [[x, y]_L, z]_t - (-1)^{|y||x|} [y, [x, z]_i]_j \right) t^{i+j} = 0;$$

i.e

$$\sum_{i \geq 0, s \geq 0} \left([x, [y, z]_i]_{s-i} - [[x, y]_L, z]_{s-i} - (-1)^{|y||x|} [y, [x, z]_i]_{s-i} \right) t^s = 0,$$

or

$$\sum_{s \geq 0} t^s \sum_{i \geq 0} \left([x, [y, z]_i]_{s-i} - [[x, y]_L, z]_{s-i} - (-1)^{|y||x|} [y, [x, z]_i]_{s-i} \right) = 0.$$

The deformation equation is equivalent to the following infinite system

$$\sum_{i=0}^s \left([x, [y, z]_i]_{s-i} - [[x, y]_L, z]_{s-i} - (-1)^{|y||x|} [y, [x, z]_i]_{s-i} \right) = 0. \quad (5.3)$$

In particular, for $s = 0$ we have $[x, [y, z]_0]_0 - [[x, y]_L, z]_0 - (-1)^{|y||x|} [y, [x, z]_0]_0$ which is the super Jacobi identity of \mathcal{N} .

The equation for $s = 1$, is equivalent to $\delta_0^2([\cdot, \dots, \cdot]_1) = 0$. Then $[\cdot, \dots, \cdot]_1$ is a 2-cocycle.

For $s \geq 2$, the identities (5.3) are equivalent to:

$$\delta^2([\cdot, \dots, \cdot]_s)(x, y, z) = - \sum_{i=1}^{s-1} \left([x, [y, z]_i]_{s-i} - [[x, y]_L, z]_{s-i} - (-1)^{|y||x|} [y, [x, z]_i]_{s-i} \right).$$

One may also prove

Theorem 5.3. *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n -ary-Nambu-Lie superalgebra and $\mathcal{N}_t = (\mathcal{N}, [\cdot, \dots, \cdot]_t)$ be a one-parameter formal deformation of \mathcal{N} , where $[\cdot, \dots, \cdot]_t = \sum_{i \geq 0} t^i [\cdot, \dots, \cdot]_i$. Then there exists an equivalent deformation $\mathcal{N}'_t = (\mathcal{N}, [\cdot, \dots, \cdot]'_t)$ where $[\cdot, \dots, \cdot]'_t = \sum_{i \geq 0} t^i [\cdot, \dots, \cdot]'_i$ such that $[\cdot, \dots, \cdot]'_i \in Z^2(\mathcal{N}, \mathcal{N})$ and doesn't belong to $B^2(\mathcal{N}, \mathcal{N})$.*

Hence, if $H^2(\mathcal{N}, \mathcal{N}) = 0$ then every formal deformation is equivalent to a trivial deformation. The n -ary-Nambu-Lie superalgebra is called rigid.

6 Cohomology of the super w_∞ 3-algebra

The generators of CHOVW algebra are given by

$$\begin{aligned} L_m^i &= (-1)^i \lambda^{i-\frac{1}{2}} z^{n+i} \frac{\partial^i}{\partial z^i}, \\ \bar{L}_m^i &= (-1)^i \lambda^{i+\frac{3}{2}} z^{n+i} \theta \frac{\partial}{\partial \theta} \frac{\partial^i}{\partial z^i}, \\ h_r^{\alpha+\frac{1}{2}} &= (-1)^{\alpha+1} \lambda^{\alpha+\frac{1}{2}} z^{r+\alpha} \frac{\partial}{\partial \theta} \frac{\partial^\alpha}{\partial z^\alpha}, \\ \bar{h}_r^{\alpha+\frac{1}{2}} &= (-1)^{\alpha+1} \lambda^{\alpha+\frac{1}{2}} z^{r+\alpha} \theta \frac{\partial^\alpha}{\partial z^\alpha}. \end{aligned}$$

The commutation relation is defined by

$$[a, b] = ab - (-1)^{|a||b|}ba. \quad (6.1)$$

Let us define a super 3-bracket as follows:

$$[a, b, c] = [a, b]c + (-1)^{|a|(|b|+|c|)}[b, c]a + (-1)^{|c|(|b|+|c|)}[c, a]b. \quad (6.2)$$

Using (6.1), (6.2) and taking the scaling limit $\lambda \rightarrow 0$, then we obtain the following super w_∞ -algebra.

$$\begin{aligned} [L_m^i, L_n^j, L_k^h] &= (h(n-m) + j(m-k) + i(k-n))L_{m+n+k}^{i+j+h-1}, \\ [L_m^i, L_n^j, \bar{L}_k^h] &= (h(n-m) + j(m-k) + i(k-n))\bar{L}_{m+n+k}^{i+j+h-1}, \\ [L_m^i, L_n^j, h_p^{\alpha+\frac{1}{2}}] &= (\alpha(n-m) + j(m-p) + i(p-n))h_{m+n+p}^{i+j+\alpha-1+\frac{1}{2}}, \\ [L_m^i, L_n^j, \bar{h}_r^{\alpha+\frac{1}{2}}] &= (\alpha(n-m) + j(m-r) + i(r-n))\bar{h}_{m+n+r}^{i+j+\alpha-1+\frac{1}{2}}, \\ [L_m^i, h_p^{\alpha+\frac{1}{2}}, \bar{h}_r^{\beta+\frac{1}{2}}] &= (i(p-r) + \alpha(r-m) + \beta(m-p))\bar{L}_{m+r+p}^{i+\alpha+\beta-1}. \end{aligned}$$

The other brackets are obtained by supersymmetry or equals 0.

This algebra is \mathbb{Z}_2 graded with

$$w_\infty = (w_\infty)_0 \oplus (w_\infty)_1 \text{ where } (w_\infty)_0 = \bigoplus_{n \in \mathbb{Z}, i \in \mathbb{N}} \langle L_n^i, \bar{L}_n^i \rangle \text{ and } (w_\infty)_1 = \bigoplus_{n \in \mathbb{Z}, i \in \mathbb{N}} \langle h_n^{i+\frac{1}{2}}, \bar{h}_n^{i+\frac{1}{2}} \rangle.$$

In the following, we describe a super w_∞ 3-algebra obtained in [1] and we compute its derivations and second cohomology group.

6.1 Derivations of the super w_∞ 3-algebra.

An even derivation D (resp. odd) is said of degree (s, t) if there exists $(s, t) \in \mathbb{Z} \times \mathbb{N}$ such that, for all $(m, i) \in \mathbb{Z} \times \mathbb{N}$, we have $D(\langle L_m^i \rangle \oplus \langle \bar{L}_m^i \rangle) \subset (\langle L_{m+s}^{i+t} \rangle \oplus \langle \bar{L}_{m+s}^{i+t} \rangle)$ and $D(\langle h_m^{i+\frac{1}{2}} \rangle \oplus \langle \bar{h}_m^{i+\frac{1}{2}} \rangle) \subset (\langle h_{m+s}^{i+t+\frac{1}{2}} \rangle \oplus \langle \bar{h}_{m+s}^{i+t+\frac{1}{2}} \rangle)$ (resp. $D(\langle L_m^i \rangle \oplus \langle \bar{L}_m^i \rangle) \subset (\langle h_{m+s}^{i+t+\frac{1}{2}} \rangle \oplus \langle \bar{h}_{m+s}^{i+t+\frac{1}{2}} \rangle)$ and $D(\langle h_m^{i+\frac{1}{2}} \rangle \oplus \langle \bar{h}_m^{i+\frac{1}{2}} \rangle) \subset (\langle L_{m+s}^{i+t} \rangle \oplus \langle \bar{L}_{m+s}^{i+t} \rangle)$).

It easy to check that $Der(w_\infty) = \bigoplus_{(s,t) \in \mathbb{Z} \times \mathbb{N}} (Der(w_\infty)_0^{(s,t)} \oplus Der(w_\infty)_1^{(s,t)})$.

Let f be a homogeneous derivation

$$f([x_1, x_2, x_3]) = [f(x_1), x_2, x_3] + (-1)^{|f||x_1|}[x_1, f(x_2), x_3] + (-1)^{|f|(|x_1|+|x_2|)}[x_1, f(x_2), f(x_3)].$$

We deduce that

$$(h(n-m) + j(m-k) + i(k-n))f(L_{m+n+k}^{i+j+h-1}) = [f(L_m^i), L_n^j, L_k^h] + [L_m^i, f(L_n^j), L_k^h] + [L_m^i, L_n^j, f(L_k^h)], \quad (6.3)$$

$$\begin{aligned} \left(h(n-m) + j(m-k) + i(k-n)\right) f(\overline{L}_{m+n+k}^{i+j+h-1}) &= \left[f(L_m^i, L_n^j, \overline{L}_k^h) + [L_m^i, f(L_n^j), \overline{L}_k^h] + [L_m^i, L_n^j, f(\overline{L}_k^h)]\right] \\ &\quad (6.4) \end{aligned}$$

$$\begin{aligned} \left(\alpha(n-m) + j(m-r) + i(r-n)\right) f(h_{m+n+r}^{i+j+\alpha-1+\frac{1}{2}}) &= \left[f(L_m^i, L_n^j, h_r^{\alpha+\frac{1}{2}}) + [L_m^i, f(L_n^j), h_r^{\alpha+\frac{1}{2}}]\right] \\ &\quad + [L_m^i, L_n^j, f(h_r^{\alpha+\frac{1}{2}})] \quad (6.5) \end{aligned}$$

$$\begin{aligned} \left(\alpha(n-m) + j(m-r) + i(r-n)\right) f(\overline{h}_{m+n+r}^{i+j+\alpha-1+\frac{1}{2}}) &= \left[f(L_m^i, L_n^j, h_r^{\alpha+\frac{1}{2}}) + [L_m^i, f(L_n^j), h_r^{\alpha+\frac{1}{2}}]\right] \\ &\quad + [L_m^i, L_n^j, f(\overline{h}_r^{\alpha+\frac{1}{2}})] \quad (6.6) \end{aligned}$$

$$\begin{aligned} \left(\alpha(m-n) + j(r-m) + i(n-r)\right) f(\overline{L}_{m+n+r}^{i+j+\alpha-1}) &= \left[f(L_m^i, h_n^{j+\frac{1}{2}}, \overline{h}_r^{\alpha+\frac{1}{2}}) + [L_m^i, f(h_n^{j+\frac{1}{2}}), \overline{h}_r^{\alpha+\frac{1}{2}}]\right] \\ &\quad + (-1)^{|f|} [L_m^i, h_n^{j+\frac{1}{2}}, f(\overline{h}_r^{\alpha+\frac{1}{2}})] \quad (6.7) \end{aligned}$$

6.1.1 Even derivations of the super w_∞ 3-algebra

Let f be an even derivation of degree (s, t) ,

$$f(L_{m,s}^{i,t}) = a_{m,s}^{i,t} L_{m+s}^{i+t} + b_{m,s}^{i,t} \overline{L}_{m+s}^{i+t},$$

$$f(\overline{L}_m^i) = c_{m,s}^{i,t} L_{m+s}^{i+t} + d_{m,s}^{i,t} \overline{L}_{m+s}^{i+t},$$

$$f(h_r^{\alpha+\frac{1}{2}}) = e_{r,s}^{\alpha,t} h_{r+s}^{\alpha+t+\frac{1}{2}} + f_{r,s}^{\alpha,t} \overline{h}_r^{\alpha+\frac{1}{2}}$$

and

$$f(\overline{h}_r^{\beta+\frac{1}{2}}) = x_{r,s}^{\beta,t} h_{r+s}^{\beta+t+\frac{1}{2}} + y_{r,s}^{\beta,t} \overline{h}_{r+s}^{\beta+t+\frac{1}{2}}.$$

By (6.3) we have

$$\begin{aligned} &a_{m+n+k,s}^{i+j+h-1,t} \left(h(n-m) + j(m-k) + i(k-n)\right) \\ &= a_{m,s}^{i,t} \left(h(n-m) + j(m-k) + i(k-n) - hs + js + t(k-n)\right) \\ &\quad + a_{n,s}^{j,t} \left(h(n-m) + (j)(m-k) + i(k-n) + hs + t(m-k) - is\right) \\ &\quad + a_{k,s}^{h,t} \left(h(n-m) + j(m-k) + i(k-n) + t(n-m) - js + is\right), \quad (6.8) \end{aligned}$$

and

$$\begin{aligned} &b_{m+n+k,s}^{i+j+h,t} \left(h(n-m) + j(m-k) + i(k-n)\right) \\ &= b_{m,s}^{i,t} \left(h(n-m) + j(m-k) + i(k-n) + t(k-n) - hs + js\right) \\ &\quad + b_{n,s}^{j,t} \left(h(n-m) + j(m-k) + i(k-n) + t(m-k) + hs - is\right) \\ &\quad + b_{k,s}^{h,t} \left(h(n-m) + j(m-k) + i(k-n) + t(n-m) - js + is\right). \quad (6.9) \end{aligned}$$

By (6.4), we have

$$\left(h(n-m) + j(m-k) + i(k-n)\right) c_{m+n+k,s}^{i+j+h-1,t} = \left(h(n-m) + j(m-k) + i(k-n) + t(n-m) - js + is\right) c_{k,s}^{h,t} \quad (6.10)$$

and

$$\begin{aligned} & \left(h(n-m) + j(m-k) + i(k-n)\right) d_{m+n+k,s}^{i+j+h-1,t} \\ = & \left(h(n-m) + j(m-k) + i(k-n) - hs + js + t(k-n)\right) a_{m,s}^{i,t} \\ & + \left(h(n-m) + j(m-k) + i(k-n) + hs + t(m-k) - is\right) a_{n,s}^{j,t} \\ & + \left(h(n-m) + j(m-k) + i(k-n) + t(n-m) - js + is\right) d_{k,s}^{h,t}. \end{aligned} \quad (6.11)$$

By (6.5) and (6.6), we obtain, exactly, the same equation as (6.11).

By (6.7), we obtain

$$\begin{aligned} & \left(\alpha(m-n) + j(r-m) + i(n-r)\right) d_{m+n+r,s}^{i+j+h-1,t} \\ = & \left(\alpha(m-n) + j(r-m) + i(n-r) - js + \alpha s + t(n-r)\right) a_{m,s}^{i,t} \\ & + \left(\alpha(m-n) + j(r-m) + i(n-r) - \alpha s + t(r-m) + is\right) e_{n,s}^{j,t} \\ & + \left(\alpha(m-n) + j(r-m) + i(n-r) + t(m-n) + js - is\right) y_{r,s}^{\alpha,t}, \end{aligned} \quad (6.12)$$

and

$$\left(\alpha(m-n) + j(r-m) + i(n-r)\right) c_{m+n+r,s}^{i+j+\alpha-1,t} = 0. \quad (6.13)$$

Taking $m = n = i = 0$, $j = 1$ (resp. $m = 1$, $n = -1$, $r = 0$, $i = 1$, $j = 0$), we obtain $c_{r,s}^{\alpha,t} = 0$, $\forall (r, s) \in \mathbb{Z} \times \mathbb{N}$.

Proposition 6.1. • *If $s + 2t \neq 0$ we have*

$$Der(w_\infty)_0^{(s,t)} = \langle ad(L_{1+s}^t, L_{-1}^1) \rangle \oplus \langle ad(L_1^0, L_{-1+s}^{1+t}) \rangle \oplus \langle ad(\bar{L}_{1+s}^t, L_{-1}^1) \rangle \oplus \langle ad(L_1^0, \bar{L}_{-1+s}^{1+t}) \rangle,$$

• *If $s + 2t = 0$ and $t \neq 0$ we have*

$$Der(w_\infty)_0^{(s,t)} = \langle ad(L_{1+s}^{1+t}, L_{-1}^0) \rangle \oplus \langle ad(L_1^1, L_{-1+s}^t) \rangle \oplus \langle ad(\bar{L}_{1+s}^{1+t}, L_{-1}^0) \rangle \oplus \langle ad(L_1^1, \bar{L}_{-1+s}^t) \rangle.$$

• *If $s + 2t = 0$ and $t = 0$ we have*

$$\begin{aligned} Der(w_\infty)_0^{(0,0)} &= \langle ad(L_{-1}^1, L_1^0) \rangle \oplus \langle ad(L_0^1, L_0^0) \rangle \oplus \langle ad(h_0^{1+\frac{1}{2}}, \bar{h}_0^{\frac{1}{2}}) \rangle \oplus \langle ad(h_1^{0+\frac{1}{2}}, \bar{h}_{-1}^{1+\frac{1}{2}}) \rangle \\ &\quad \oplus \langle \varphi_1 \rangle \oplus \langle \varphi_2 \rangle, \end{aligned}$$

where

$$\begin{aligned} \varphi_1(L_k^h) &= \varphi_1(h_k^{h+\frac{1}{2}}) = 0, \quad \varphi_1(\bar{L}_k^h) = \bar{L}_k^h, \quad \varphi_1(\bar{h}_k^{\alpha+\frac{1}{2}}) = \bar{h}_k^{\alpha+\frac{1}{2}}. \\ \varphi_2(L_k^h) &= \varphi_2(\bar{h}_k^{\alpha+\frac{1}{2}}) = 0, \quad \varphi_2(\bar{L}_k^h) = \bar{L}_k^h, \quad \varphi_2(h_k^{\alpha+\frac{1}{2}}) = h_k^{\alpha+\frac{1}{2}}. \end{aligned}$$

Proof. Taking $m = 1$, $n = -1$, $i = 0$, $j = 1$ in (6.3) we obtain

$$\begin{aligned} (-2h + 1 - k)f(L_k^h) &= a_{1,s}^{0,t}ad(L_{1+s}^t, L_{-1}^1)(L_k^h) + b_{1,s}^{0,t}ad(\overline{L}_{1+s}^t, L_{-1}^1)(L_k^h) \\ &\quad + a_{-1,s}^{1,t}ad(L_1^0, L_{-1+s}^{1+t})(L_k^h) + b_{-1,s}^{1,t}ad(L_1^0, \overline{L}_{-1+s}^{1+t})(L_k^h) \\ &\quad + (-2h - 2t + 1 - k - s)f(L_k^h), \end{aligned}$$

which gives (if $s + 2t \neq 0$)

$$\begin{aligned} f(X_k^h) &= \frac{a_{1,s}^{0,t}}{s + 2t}ad(L_{1+s}^t, L_{-1}^1)(X_k^h) + \frac{b_{1,s}^{0,t}}{s + 2t}ad(\overline{L}_{1+s}^t, L_{-1}^1)(X_k^h) \\ &\quad + \frac{a_{-1,s}^{1,t}}{s + 2t}ad(L_1^0, L_{-1+s}^{1+t})(X_k^h) + \frac{b_{-1,s}^{1,t}}{s + 2t}ad(L_1^0, \overline{L}_{-1+s}^{1+t})(X_k^h) \text{ for any } X_k^h \in w_\infty \end{aligned}$$

Taking $m = 1$, $n = -1$, $i = 1$, $j = 0$ in (6.3) we obtain

$$\begin{aligned} (-2h + 1 + k)f(X_k^h) &= a_{1,s}^{1,t}ad(L_{1+s}^{1+t}, L_{-1}^0)(X_k^h) + b_{1,s}^{1,t}ad(\overline{L}_{1+s}^{1+t}, L_{-1}^0)(X_k^h) \\ &\quad + a_{-1,s}^{0,t}ad(L_1^1, L_{-1+s}^t)(X_k^h) + b_{-1,s}^{0,t}ad(L_1^1, \overline{L}_{-1+s}^t)(X_k^h) \\ &\quad + (-2h - 2t + 1 + k + s)f(X_k^h), \end{aligned}$$

which gives

$$\begin{aligned} f(X_k^h) &= \frac{a_{1,s}^{1,t}}{4t}ad(L_{1+s}^{1+t}, L_{-1}^0)(X_k^h) + \frac{b_{1,s}^{0,t}}{4t}ad(\overline{L}_{1+s}^{1+t}, L_{-1}^0)(X_k^h) \\ &\quad + \frac{a_{-1,s}^{0,t}}{4t}ad(L_1^1, L_{-1+s}^t)(X_k^h) + \frac{b_{-1,s}^{1,t}}{4t}ad(L_1^1, \overline{L}_{-1+s}^t)(X_k^h) \text{ if } s + 2t = 0 \text{ and } t \neq 0. \end{aligned}$$

Taking $s = 0$, $t = 0$, $m = 0$, $n = 0$, $i = 2$, $j = 0$ in (6.8), we obtain

$$a_{k,0}^{h,0} = ha_{0,0}^{2,0} + ha_{0,0}^{0,0} + a_{k,0}^{0,0}, \forall k \in \mathbb{Z}^*. \quad (6.14)$$

Taking $s = 0$, $t = 0$, $h = 0$, $m = 1$, $i = 0$, $n = 0$, $j = 1$ in (6.8), we obtain

$$(1 - k)a_{k+1,0}^{0,0} = (1 - k)a_{1,0}^{0,0} + (1 - k)a_{0,0}^{1,0} + (1 - k)a_{k,0}^{0,0}.$$

We deduce that

$$a_{k,0}^{0,0} = (k - 1)a_{1,0}^{0,0} + (k - 1)a_{0,0}^{1,0} + a_{1,0}^{0,0} \quad \forall k \in \mathbb{Z}^*.$$

Then $a_{k,0}^{h,0} = ha_{0,0}^{2,0} + ha_{0,0}^{0,0} + (k - 1)a_{1,0}^{0,0} + (k - 1)a_{0,0}^{1,0} + a_{1,0}^{0,0}$. So,

$$a_{1,0}^{2,0} = 2a_{0,0}^{2,0} + 2a_{0,0}^{0,0} + a_{1,0}^{0,0} = 2a_{0,0}^{2,0} - 2a_{0,0}^{1,0} + a_{1,0}^{0,0}$$

Taking $m = 1$, $n = 0$, $k = 0$, $i = 0$, $j = 1$, $h = 2$ in (6.8), we obtain

$$a_{1,0}^{2,0} = a_{0,0}^{1,0} + a_{1,0}^{0,0} + a_{0,0}^{2,0}$$

So

$$a_{0,0}^{2,0} = 3a_{0,0}^{1,0}.$$

We deduce that

$$a_{k,0}^{h,0} = (2h + k - 1)a_{0,0}^{1,0} + ka_{1,0}^{0,0}. \quad (6.15)$$

Then

$$b_{k,0}^{h,0} = (2h + k - 1)b_{0,0}^{1,0} + kb_{1,0}^{0,0}.$$

Taking $s = 0$, $t = 0$, $m = 1$, $n = -1$, $i = j = 1$ in (6.12), we obtain

$$d_{k,0}^{h,0} = (h - 2)(a_{1,0}^{1,0} + a_{-1,0}^{1,0}) + d_{k,0}^{2,0}$$

Since $a_{1,0}^{1,0} + a_{-1,0}^{1,0} = a_{1,0}^{0,0}$ we deduce that $d_{k,0}^{h,0} = 2(h - 2)a_{0,0}^{1,0} + d_{k,0}^{2,0}$.

Taking $s = 0$, $t = 0$, $m = 1$, $n = -0$, $i = 0$, $j = 1$, $h = 2$ in (6.12) we obtain

$$-(k + 1)d_{k+1,0}^{2,0} = -(k + 1)a_{1,0}^{0,0} - (k + 1)a_{0,0}^{1,0} - (k + 1)d_{k,0}^{3,0}.$$

One can deduce that

$$d_{k,0}^{h,0} = (2h + k - 1)a_{0,0}^{1,0} + ka_{1,0}^{0,0} + d_{0,0}^{2,0} - 3a_{0,0}^{1,0}, \quad \forall k \in \mathbb{Z} \setminus \{-1\}. \quad (6.16)$$

Then,

$$e_{k,0}^{h,0} = (2h + k - 1)a_{0,0}^{1,0} + ka_{1,0}^{0,0} + e_{0,0}^{2,0} - 3a_{0,0}^{1,0}, \quad \forall k \in \mathbb{Z} \setminus \{-1\}, \quad (6.17)$$

and

$$y_{r,0}^{\alpha,0} = (2\alpha + r - 1)a_{0,0}^{1,0} + ra_{1,0}^{0,0} + y_{0,0}^{2,0} - 3a_{0,0}^{1,0}, \quad \forall r \in \mathbb{Z} \setminus \{-1\}. \quad (6.18)$$

Taking $m = 1$, $n = -1$, $i = j = 1$ in (6.5), we obtain $x_{r,0}^{\alpha,0} = x_k^2 = 0$, $\forall h \in \mathbb{N}^* \setminus \{1\}$.

So, $f_{r,0}^{\alpha,0} = f_r^2 = 0$, $\forall \alpha \in \mathbb{N}^* \setminus \{1\}$.

Using (6.15), (6.16), (6.17) and (6.18) in (6.12), we obtain $d_{0,0}^{2,0} = -3a_{0,0}^{1,0} + e_{0,0}^{2,0} + y_{0,0}^{2,0}$.

Therefore,

$$f = a_{0,0}^{1,0}ad(L_{-1}^1, L_1^0) + a_{1,0}^{0,0}ad(L_0^1, L_0^0) + b_{0,0}^{1,0}ad(h_1^0, \bar{h}_{-1}^1) - b_{1,0}^{0,0}ad(h_0^1, \bar{h}_0^0) + (d_{0,0}^{2,0} - 3a_{0,0}^{1,0})\varphi_1 + (e_{0,0}^{2,0} - 3a_{0,0}^{1,0})\varphi_2,$$

where,

$$\begin{aligned} \varphi_1(L_k^h) &= \varphi_1(h_k^{h+\frac{1}{2}}) = 0, \quad \varphi_1(\bar{L}_k^h) = \bar{L}_k^h, \quad \varphi_1(\bar{h}_k^{\alpha+\frac{1}{2}}) = \bar{h}_k^{\alpha+\frac{1}{2}}, \\ \varphi_2(L_k^h) &= \varphi_2(\bar{h}_k^{\alpha+\frac{1}{2}}) = 0, \quad \varphi_2(\bar{L}_k^h) = \bar{L}_k^h, \quad \varphi_2(h_k^{\alpha+\frac{1}{2}}) = h_k^{\alpha+\frac{1}{2}}. \end{aligned}$$

□

6.2 Odd derivation

Let f be an odd derivation of degree (s, t) ,

$$\begin{aligned} f(L_{m,s}^{i,t}) &= a_{m,s}^{i,t} h_{m+s}^{i+t+\frac{1}{2}} + b_{m,s}^{i,t} \bar{h}_{m+s}^{i+t+\frac{1}{2}}, \\ f(\bar{L}_m^i) &= c_{m,s}^{i,t} h_{m+s}^{i+t} + d_{m,s}^{i,t} \bar{h}_{m+s}^{i+t}, \\ f(h_r^{\alpha+\frac{1}{2}}) &= e_{r,s}^{\alpha,t} l_{r+s}^{\alpha+t} + f_{r,s}^{\alpha,t} \bar{L}_{r+s}^{\alpha+t} \\ f(\bar{h}_r^{\beta+\frac{1}{2}}) &= x_{r,s}^{\beta,t} l_{r+s}^{\beta+t} + y_{r,s}^{\beta,t} \bar{L}_{r+s}^{\beta+t} \end{aligned} \quad (6.19)$$

Proposition 6.2. • If $s + 2t \neq 0$, then we have

$$\begin{aligned} &Der(w_\infty)_1^{(s,t)} \\ &= \langle ad(h_{1+s}^{t+\frac{1}{2}}, L_{-1}^1) \rangle \oplus \langle ad(\bar{h}_{1+s}^{t+\frac{1}{2}}, L_{-1}^1) \rangle \oplus \langle ad(L_1^0, h_{-1+s}^{1+t+\frac{1}{2}}) \rangle \oplus \langle ad(L_1^0, \bar{h}_{-1+s}^{1+t+\frac{1}{2}}) \rangle. \end{aligned}$$

• If $s + 2t = 0$ and $t \neq 0$ we have

$$\begin{aligned} &Der(w_\infty)_1^{(s,t)} \\ &= \langle ad(h_{1+s}^{1+t+\frac{1}{2}}, L_{-1}^0) \rangle \oplus \langle ad(\bar{h}_{1+s}^{1+t+\frac{1}{2}}, L_{-1}^0) \rangle \oplus \langle ad(L_1^1, h_{-1+s}^{t+\frac{1}{2}}) \rangle \oplus \langle ad(L_1^1, \bar{h}_{-1+s}^{t+\frac{1}{2}}) \rangle. \end{aligned}$$

• If $s + 2t = 0$ and $t = 0$

$$Der(w_\infty)_1^{(0,0)} = \langle ad(L_1^0, h_{-1}^{1+\frac{1}{2}}) \rangle \oplus \langle ad(L_1^0, \bar{h}_{-1}^{1+\frac{1}{2}}) \rangle \oplus \langle ad(L_0^1, h_0^{1+\frac{1}{2}}) \rangle \oplus \langle ad(L_0^1, \bar{h}_0^{1+\frac{1}{2}}) \rangle$$

Proof. Setting $m = 1$, $n = -1$, $i = 0$, $j = 1$ in (6.3), one has

$$\begin{aligned} (-2h + 1 - k)f(L_k^h) &= a_{1,s}^{0,t} ad(h_{1+s}^{t+\frac{1}{2}}, L_{-1}^1)(L_k^h) + b_{1,s}^{0,t} ad(\bar{h}_{1+s}^{t+\frac{1}{2}}, L_{-1}^1)(L_k^h) \\ &\quad + a_{-1,s}^{1,t} ad(L_1^0, h_{-1+s}^{1+t+\frac{1}{2}})(L_k^h) + b_{-1,s}^{1,t} ad(L_1^0, \bar{h}_{-1+s}^{1+t+\frac{1}{2}})(L_k^h) \\ &\quad + (-2h - 2t + 1 - k - s)f(L_k^h), \end{aligned}$$

which gives

$$\begin{aligned} f(L_k^h) &= \frac{a_{1,s}^{0,t}}{s+2t} ad(h_{1+s}^{t+\frac{1}{2}}, L_{-1}^1)(L_k^h) + \frac{b_{1,s}^{0,t}}{s+2t} ad(\bar{h}_{1+s}^{t+\frac{1}{2}}, L_{-1}^1)(L_k^h) \\ &\quad + \frac{a_{-1,s}^{1,t}}{s+2t} ad(L_1^0, h_{-1+s}^{1+t+\frac{1}{2}})(L_k^h) + \frac{b_{-1,s}^{1,t}}{s+2t} ad(L_1^0, \bar{h}_{-1+s}^{1+t+\frac{1}{2}})(L_k^h) \text{ if } s+2t \neq 0. \end{aligned}$$

Furthermore, taking $m = 1$, $n = -1$, $i = 1$, $j = 0$ in (6.3), one has

$$\begin{aligned} f(L_k^h) &= \frac{a_{1,s}^{1,t}}{4t} ad(h_{1+s}^{1+t+\frac{1}{2}}, L_{-1}^0)(L_k^h) + \frac{b_{1,s}^{0,t}}{4t} ad(\bar{h}_{1+s}^{1+t+\frac{1}{2}}, L_{-1}^0)(L_k^h) \\ &\quad + \frac{a_{-1,s}^{0,t}}{4t} ad(L_1^1, h_{-1+s}^{t+\frac{1}{2}})(L_k^h) + \frac{b_{-1,s}^{1,t}}{4t} ad(L_1^1, \bar{h}_{-1+s}^{t+\frac{1}{2}})(L_k^h) \text{ if } t \neq 0. \end{aligned}$$

By (6.3) and (6.19) we obtain, exactly, the same equation as (6.8) and (6.9).
 By (6.4) and (6.19) we obtain, exactly, the same equation as (6.10) and (6.11).
 By (6.5) and (6.19) we obtain (6.11) and (6.10).
 By (6.6) and (6.19) we obtain (6.11) and (6.10).
 Therefore,

$$\begin{aligned} a_{k,0}^{h,0} &= (2h+k-1)a_{0,0}^{1,0} + ka_{1,0}^{0,0} \\ b_{k,0}^{h,0} &= (2h+k-1)b_{0,0}^{1,0} + kb_{1,0}^{0,0} \\ d_{k,0}^{h,0} &= (2h+k-1)a_{0,0}^{1,0} + ka_{1,0}^{0,0} + d_{0,0}^{2,0} - 3a_{0,0}^{1,0}, \quad \forall k \in \mathbb{Z} \setminus \{-1\}. \end{aligned}$$

We deduce that

$$f = a_{0,0}^{1,0}ad(L_1^0, h_{-1}^{1+\frac{1}{2}}) + a_{1,0}^{0,0}ad(L_1^0, \bar{h}_{-1}^{1+\frac{1}{2}}) + b_{0,0}^{1,0}ad(L_0^0, h_0^{1+\frac{1}{2}}) + b_{1,0}^{0,0}ad(L_0^0, \bar{h}_0^{1+\frac{1}{2}})$$

□

6.3 Cohomology space $H_0^2(w_\infty, \mathbb{C})$ of w_∞

In the following we describe the cohomology space $H_0^2(w_\infty, \mathbb{C})$. We denote by $[f]$ the cohomology class of an element f .

Theorem 6.3. *The second cohomology of the super w_∞ 3-algebra with values in the trivial module vanishes, i.e*

$$H_0^2(w_\infty, \mathbb{C}) = \{0\}.$$

Proof. For all $f \in Z_0^1(w_\infty, \mathbb{C})$ (resp. $f \in Z_0^2(w_\infty, C)$) we have, respectively,

$$\delta^1(f)(x_0, z) = f([x_1, x_2, x_3]) = 0, \quad (6.20)$$

$$\delta^2(f)(x_0, x_1, z) = -f([x_0, x_1], z) - (-1)^{|x_0||x_1|}f(x_1, ad(x_0)(z)) + f(x_0, ad(x_1)(z)) = 0. \quad (6.21)$$

$$\textbf{Case 1 } (X_p^v, X_q^l, X_k^h) \in \left\{ (L_m^i, L_n^j, L_k^h), (L_m^i, L_n^j, \bar{L}_k^h), (L_m^i, L_n^j, h_r^{\alpha+\frac{1}{2}}), (L_m^i, L_n^j, \bar{h}_r^{\alpha+\frac{1}{2}}), (h_r^{\alpha+\frac{1}{2}}, L_m^i, \bar{h}_s^{\beta+\frac{1}{2}}) \right\}.$$

Using (6.21), we have

$$-f([L_m^i \wedge L_n^j, X_p^v \wedge X_q^l], X_k^h) - f(X_p^v \wedge X_q^l, ad(L_m^i \wedge L_n^j)(X_k^h)) + f(L_m^i \wedge L_n^j, ad(X_p^v \wedge X_q^l)(X_k^h)) = 0.$$

Since $[x, y]_l = \sum_{i=1}^{n-1} (-1)^{|x|(|y_1|+\dots+|y_{i-1}|)} y_1 \wedge \dots \wedge ad(x)(y_i) \wedge \dots \wedge y_{n-1}$, and $ad(a \wedge b)(z) = [a, b, z]$

then

$$\begin{aligned} &-f([L_m^i \wedge L_n^j, X_p^v] \wedge X_q^l, X_k^h) - f(X_p^v, [L_m^i \wedge L_n^j, X_q^l], X_k^h) \\ &-f(X_p^v \wedge X_q^l, [L_m^i \wedge L_n^j, X_k^h]) + f(L_m^i \wedge L_n^j, [X_p^v \wedge X_q^l, X_k^h]) = 0. \end{aligned}$$

So,

$$\begin{aligned}
& -\left(v(n-m) + j(m-\bar{p}) + i(\bar{p}-n)\right) f\left(X_{m+n+p}^{i+j+v-1} \wedge X_q^l, X_k^h\right) \\
& -\left(\bar{l}(n-m) + j(m-\bar{q}) + i(\bar{q}-n)\right) f\left(X_{\bar{p}}^v \wedge X_{m+n+\bar{q}}^{i+j+l-1}, X_k^h\right) \\
& -\left(\bar{h}(n-m) + j(m-k) + i(k-n)\right) f\left(X_p^v \wedge X_q^l, X_{m+n+k}^{i+j+h-1}\right) \\
& +\left(\bar{h}(\bar{q}-\bar{p}) + l(\bar{p}-k) + i(k-\bar{q})\right) f\left(L_m^i \wedge L_n^j, X_{p+q+k}^{v+l+h-1}\right) = 0,
\end{aligned} \tag{6.22}$$

where \bar{p} is the integer part of p .

Setting $m = 0$, $n = 0$, $i = 1$, $j = 0$, in (6.22), we obtain

$$f(X_p^v \wedge X_q^l, X_k^h) = \frac{h(q-p) + l(p-k) + (k-q)}{p+q+k} f(L_0^1 \wedge L_0^0, X_{p+q+k}^{v+t+h-1}) \quad (p+q+k \neq 0).$$

Setting $m = 1$, $n = -1$, $i = 1$, $j = 0$, $k = -p-q$ (6.22), we obtain

$$f(X_p^v \wedge X_q^l, X_{-p-q}^h) = -\frac{h(q-p) + l(2p-q) + (-p-2q)}{2l+2v+2h-3} f(L_1^1 \wedge L_{-1}^0, X_0^{v+l+h-1}).$$

Setting $m = 1$, $j = 0$, $i = 1$, $p = 1$, $q = -1$, $v = 1$, $l = 0$, $k = 0$ (6.22), we obtain

$$(-2h-1)f(L_1^1 \wedge L_{-1}^0, X_0^h) = 0.$$

Then $f(L_1^1 \wedge L_{-1}^0, X_0^h) = 0$.

We deduce that

$$f(X_p^v \wedge X_q^l, X_{-p-q}^h) = 0.$$

We define an endomorphism g of w_∞ by $g(X_k^h) = -\frac{1}{k}f(L_1^1 \wedge L_{-1}^0, X_k^h)$ and $g(X_0^h) = 0$.

By (6.20) we obtain

$$\delta^1(g)(X_p^v \wedge X_q^l, X_k^h) = -g(ad(X_p^v \wedge X_q^l)(X_k^h)) = f(L_m^i \wedge L_n^j, X_k^h).$$

Case 2 $(X_p^v, X_q^l, X_k^h) \notin \left\{ (L_m^i, L_n^j, L_k^h), (L_m^i, L_n^j, \bar{L}_k^h), (L_m^i, L_n^j, h_r^{\alpha+\frac{1}{2}}), (L_m^i, L_n^j, \bar{h}_r^{\alpha+\frac{1}{2}}), (h_r^{\alpha+\frac{1}{2}}, L_m^i, \bar{h}_s^{\beta+\frac{1}{2}}) \right\}$.

Using (6.21), we have

$$\begin{aligned}
& -\left(v(n-m) + j(m-p) + i(p-n)\right) f\left(X_{m+n+p}^{i+j+v-1} \wedge X_q^l, X_k^h\right) \\
& -\left(l(n-m) + j(m-q) + i(q-n)\right) f\left(X_p^v \wedge X_{m+n+q}^{i+j+l-1}, X_k^h\right) \\
& -\left(h(n-m) + j(m-k) + i(k-n)\right) f\left(X_p^v \wedge X_q^l, X_{m+n+k}^{i+j+h-1}\right) \\
& +f\left(L_m^i \wedge L_n^j, 0\right) = 0.
\end{aligned} \tag{6.23}$$

Setting $m = 0$, $n = 0$, $i = 1$, $j = 0$, in (6.23), we obtain $f(X_p^v \wedge X_q^l, X_k^h) = 0$ ($p+q+k \neq 0$).

Setting $m = 1$, $n = -1$, $i = 1$, $j = 0$, in (6.23), we obtain $f(X_p^v \wedge X_q^l, X_k^h) = 0$.

□

6.4 Second cohomology $H^2(w_\infty, w_\infty)$.

In this section, we aim compute the second cohomology group of w_∞ with values in itself.

Theorem 6.4. *The second cohomology of the 3-ary-Nambu-Lie superalgebra w_∞ with values in the adjoint module vanishes, i.e.*

$$H^2(w_\infty, w_\infty) = \{0\}.$$

Then, every formal deformation is equivalent to trivial deformation.

Proof. For all $g \in Z_0^1(w_\infty)$ (resp. $f \in Z_{0,s}^2(w_\infty)$) we have respectively

$$\begin{aligned} & \delta^1(g)(L_p^v \wedge L_q^l, X_k^h) \\ &= -g([L_p^v \wedge L_q^l, X_k^h]) + [L_p^v \wedge L_q^l, g(X_k^h)] - [X_k^h \wedge L_q^l, g(L_p^v)] + [X_k^h \wedge L_p^v, g(L_q^l)] = 0. \end{aligned}$$

$$\begin{aligned} & \delta^2(f)(L_m^i \wedge L_n^j, X_p^v \wedge X_q^l, X_k^h) \\ &= -f([L_m^i \wedge L_n^j, X_p^v] \wedge X_q^l, X_k^h) - f(X_p^v, [L_m^i \wedge L_n^j, X_q^l], X_k^h) - f(X_p^v \wedge X_q^l, [L_m^i \wedge L_n^j, X_k^h]) \\ & \quad + f(L_m^i \wedge L_n^j, [X_p^v \wedge X_q^l, X_k^h]) + [L_m^i \wedge L_n^j, f(X_p^v \wedge X_q^l, X_k^h)] - [X_p^v \wedge X_q^l, f(L_m^i \wedge L_n^j, X_k^h)] \\ & \quad + [X_k^h \wedge X_q^l, f(L_m^i \wedge L_n^j, X_p^v)] - [X_k^h \wedge X_p^v, f(L_m^i \wedge L_n^j, X_q^l)] = 0. \end{aligned}$$

Case 1: $s \neq 0$.

Setting $m = 0$, $n = 0$, $i = 0$, $j = 1$, in (6.22), we obtain

$$\begin{aligned} & (p + q + k)f(X_p^v \wedge X_q^l, X_k^h) \\ & + (h(q - p) + l(p - k) + v(k - q))f(L_0^0 \wedge L_0^1, X_{p+q+k}^{v+l+h-1}) + [L_0^0 \wedge L_0^1, f(X_p^v \wedge X_q^l, X_k^h)] \\ & - [X_p^v \wedge X_q^l, f(L_0^0 \wedge L_0^1, X_k^h)] + [X_k^h \wedge X_q^l, f(L_0^0 \wedge L_0^1, X_p^v)] - [X_k^h \wedge X_p^v, f(L_0^0 \wedge L_0^1, X_q^l)] = 0. \end{aligned}$$

We denote by g the linear map defined on w_∞ by $g(X_k^h) = -\frac{1}{s}f(L_0^0 \wedge L_0^1, X_k^h)$.

It easy to verify that $\delta^1(g)(X_p^v \wedge X_q^l, X_k^h) = f(X_p^v \wedge X_q^l, X_k^h)$.

Case 2: $s = 0$.

Setting $m = 1$, $n = -1$, $i = 1$, $j = 0$ in (6.22), we obtain

$$\begin{aligned} & -2f(X_p^v \wedge X_q^l, X_k^h) + (h(q - p) + l(p - k) + v(k - q))f(L_1^1 \wedge L_{-1}^0, X_{p+q+k}^{v+l+h-1}) \\ & - [X_p^v \wedge X_q^l, f(L_1^1 \wedge L_{-1}^0, X_k^h)] + [X_k^h \wedge X_q^l, f(L_1^1 \wedge L_{-1}^0, X_p^v)] - [X_k^h \wedge X_p^v, f(L_1^1 \wedge L_{-1}^0, X_q^l)] = 0 \end{aligned}$$

We denote by g the linear map defined on w_∞ by $g(X_k^h) = -\frac{1}{2}f(L_1^1 \wedge L_{-1}^0, X_k^h)$.

It easy to verify that $\delta^1(g)(X_p^v \wedge X_q^l, X_k^h) = f(X_p^v \wedge X_q^l, X_k^h)$, that ends the proof. \square

References

- [1] Min-Rue Chen, Ke Wua and Wei-Zhong Zhao: *Super w1 3-algebra*, Sissa-Springer **090**, no. 10, 1007 (2011).
- [2] P. D. Beites and A. P. Pozhidaev: *ON SIMPLE FILIPPOV SUPERALGEBRAS OF TYPE $A(m, n)$* , arxiv.: **1008.0715v1**, , 1007 (2010).
- [3] Ruipu Bai and Ying Li: *T_θ^* -Extensions of n -Lie Algebras*, International Scholarly Research Network ISRN Algebra: **10.5402**,381875 , (2011).
- [4] Faouzi Ammar, Abdenacer Makhlouf, Nejib Saadoui: *Cohomology of Hom-Lie superalgebras and q -deformed Witt superalgebra*, arXiv: **1204.6244**, (2012).
- [5] Nambu Y: *Generalized Hamiltonian mechanics*, Phys. Rev. D7: **17405-2412**, (1973).
- [6] Bagger J. and Lambert N.: *Gauge Symmetry and Supersymmetry of multiple M2Branes*, arXiv:0711.0955v2[hep-th] (2007) Phys. Rev. D77: **065008**, (2008).
- [7] Basu A. and Harvey J.A.: *The M2-M5 brane system and a generalised nahm equation* Nucl, Phys. B713: (2005).
- [8] Kerner R. : *Ternary algebraic structures and their application in physics, in the "Proc. BTLP 23rd international Colloquium on Group Theoretical Methods in Physics"* arXiv math-ph:**0011023**, (2000)
- [9] R. Kerner, Z3-graded algebras and non-commutative gauge theories, dans le livre, in: Z. Oziewicz, B. Jancewicz, A. Borowiec (Eds.), *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, Kluwer Academic Publishers, 1993, pp. 349357.
- [10] ——— : *The cubic chessboard : Geometry and physics*, *Classical Quantum Gravity* 14, A203-A225 (1997).
- [11] Daletskii Y.L. and Takhtajan L.A.: *Leibniz and Lie Structures for Nambu algebra*, Letters in Mathematical Physics **39**, (1997) 127141.
- [12] Takhtajan L. : *TO n foundation of the generalized Nambu mechanics*, Comm. Math. Phys. **160**(1994), 295-315.
- [13] Filippov V., *n -Lie algebras*, Sibirsk. Mat. Zh. **26**, 126-140 (1985) (English transl.: Siberian Math. J. 26, 879-891 (1985)).
- [14] Cassas J.M., Loday J.-L. and Pirashvili *Leibniz n -algebras*, Forum Math. **14** (2002), 189207.
- [15] Ataguema H., Makhlouf A. *Deformations of ternary algebras*, Journal of Generalized Lie Theory and Applications, vol. 1, (2007), 4145.
- [16] ——— *Notes on cohomologies of ternary algebras of associative type*, Journal of Generalized Lie Theory and Applications 3 no. 3, (2009), 157174

- [17] Gautheron P.: *Some Remarks Concerning Nambu Mechanics*, Letters in Mathematical Physics **37** (1996) 103116.
- [18] J.A. De Azcarraga, J.M. Izquierdo, Cohomology of Filippov algebras and an analogue of Whiteheads lemma, J. Phys. Conf. Ser. 175 (2009) 012001
- [19] Takhtajan L., :*On foundation of the generalized Nambu mechanics*, Comm. Math. Phys. **160** (1994), 295-315.
- [20] L. Takhtajan. *A higher order analog of Chevally-Eilenberg complex and deformation theory of n -algebras*, St. Petersburg Math. J., 6 (1995), 429-438.